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Asymptotic Behavior of the Quantum Harmonic Oscillator Driven by a Random Time-Dependent Electric Field

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This paper investigates the evolution of the state vector of a charged quantum particle in a harmonic oscillator driven by a time-dependent electric field. The external field randomly oscillates and its amplitude is small but it acts long enough so that we can solve the problem in the asymptotic framework corresponding to a field amplitude which tends to zero and a field duration which tends to infinity. We describe the effective evolution equation of the state vector, which reads as a stochastic partial differential equation. We explicitly describe the transition probabilities, which are characterized by a polynomial decay of the probabilities corresponding to the low-energy eigenstates, and give the exact statistical distribution of the energy of the particle.

KEY WORDS: Harmonic oscillator; random perturbations; asymptotic analysis; stochastic calculus.

1. INTRODUCTION

The quantum harmonic oscillator has been extensively studied, not only because it is a system that can be exactly solved and a great pedagogical tool, but it is also a very relevant system.[1] Indeed a lot of systems close to a stable equilibrium can be described by an oscillator or a collection of decoupled harmonic oscillators. Furthermore time-independent and time-dependent modifications of this model have been investigated, handling by the well-known perturbation theory. Literature contains a lot of applications

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and discussions of special types of perturbations: sudden, adiabatic, periodic,....[2] The considered phenomena are described by the Hamiltonian:

$$H(t) = H^0 + H^1(t), \tag{1.1}$$

where H^0 is the time-independent Hamiltonian of the harmonic oscillator, whose eigenvalue problem has been solved, and H^1 is a small time-dependent perturbation. The typical question one asks is the following. If at $t = 0$ the system is in the eigenstate ψ^0 of H^0, what is the probability for it to be in some given eigenstate? Most results that have been obtained follow a scheme in which the answers are computed in a perturbation series in powers of H^1.[1,2] Indeed it is exceptional to find closed-form expressions, except for some very particular types of perturbations.[3] Nevertheless rigorous results have been obtained for time-dependent perturbations of the harmonic oscillator. Most of them concern periodic driven force.[4–6] Although the problem is far less understood in the case of random perturbations, literature contains some results about systems with randomly time-dependent external driving force. A general class of quantum systems in Markovian potentials has been treated in detail.[7,8] Under suitable conditions on the dynamics of the random potential, it is shown in ref. 9 that the spectrum of the quasi-energy operator is continuous. In ref. 10 the authors study the long-time stability of oscillators driven by time-dependent forces originating from dynamical systems with varying degrees of randomness and focus on the asymptotic energy growth. In this paper we consider a charged particle in a harmonic oscillator which is driven by a weak random time-dependent electric field. We aim at studying this problem by a rigorous and non-perturbative method. Our approach is inspired by the works of Papanicolaou and its co-authors about waves in random media.[11,12] The first step consists in determining the characteristic scales of the problem at hand: oscillation frequency of the harmonic oscillator, amplitude, coherence time and duration of the random perturbations. We then study the asymptotic evolution of the state vector in the asymptotic framework based on the separation of these scales. Our main aim is to exhibit the asymptotic regime which corresponds to the case where the amplitudes of the random fluctuations go to zero and the duration of the external field goes to infinity. We then describe explicitly the effective random evolution of the state vector and the probability transitions. The paper is organized as follows. In Section 2 we review the main features of the harmonic oscillator, while we state our main convergence result about the effective evolution of the state vector of the particle in Section 3. Before proving this result in Section 5, we give remarkable properties of the asymptotic system in Section 4.

2. THE HARMONIC OSCILLATOR

2.1. The Main Equation

We consider the quantum oscillator, that is to say, a particle of mass M whose state vector in the coordinate basis obeys the Schrödinger equation:[1]

$$i\hbar \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2M} \Delta\psi + \frac{1}{2} M\omega^2(x^2+y^2+z^2)\,\psi \tag{2.1}$$

where Δ is the Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ and ω is the oscillation frequency. In order to transform this equation into a standard and dimensionless form, we multiply the spatial coordinates x, y, z by $r_0^{-1} := (M\omega/\hbar)^{1/2}$ and the time t by to $t_0^{-1} := \omega$, so that (2.1) now reads:

$$2i \frac{\partial\psi}{\partial t} = -\Delta\psi + (x^2+y^2+z^2)\,\psi \tag{2.2}$$

2.2. Eigenvalues and Eigenstates

The spectrum of the harmonic oscillator is pure point with state energies $(2p+2q+2r+3)/2$ and corresponding eigenstates:[1]

$$f_{p,q,r}(x, y, z) = f_p(x)\, f_q(y)\, f_r(z) \tag{2.3}$$

where the real-valued functions f_p are the so-called Hermite–Gaussian functions:

$$f_p(x) = \frac{1}{\sqrt{2p\sqrt{\pi}p!}} H_p(x)\, e^{-x^2/2}, \qquad H_p(x) = (-1)^p e^{x^2} \frac{d^p}{dx^p} e^{-x^2} \tag{2.4}$$

The family $(f_{p,q,r})_{p,q,r\in\mathbb{N}}$ is complete in the following sense.[13]

Proposition 2.1. 1. The $(f_{p,q,r})_{p,q,r\in\mathbb{N}}$ are an orthonormal and complete set in $L^2(\mathbb{R}^3, \mathbb{C})$:

$$\int_{\mathbb{R}^3} f_{p,q,r}(x, y, z)\, f_{p',q',r'}(x, y, z)\, dx\, dy\, dz = \delta_{pp'}\,\delta_{qq'}\,\delta_{rr'} \tag{2.5}$$

where δ stands for the Kronecker's symbol.

2. $(t, x, y, z) \mapsto e^{-i((2p+2q+2r+3)/2)\,t} f_{p,q,r}(x, y, z)$ is a solution of (2.2) for any $p, q, r \in \mathbb{N}$.

We define the eigenstate decomposition as the map $\Theta: \psi \in L^2(\mathbb{R}^3, \mathbb{C}) \mapsto (c_{p,q,r})_{p,q,r \in \mathbb{N}}$, where $c_{p,q,r}$ is defined by:

$$\Theta(\psi)_{p,q,r} := c_{p,q,r} = \int_{\mathbb{R}^3} f_{p,q,r}(x, y, z)\, \psi(x, y, z)\, dx\, dy\, dz \tag{2.6}$$

By Proposition 2.1, Θ is an isometry from $L^2(\mathbb{R}^3, \mathbb{C})$ onto l^2, the space of all the sequences $(c_{p,q,r})_{p,q,r \in \mathbb{N}}$ from $\mathbb{N}^3$ into $\mathbb{C}$ which are squared integrable. l^2 is equipped with its usual scalar product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$:

$$\langle b, c \rangle = \sum_{p,q,r=0}^{\infty} b^*_{p,q,r} c_{p,q,r}, \qquad \|c\|^2 = \sum_{p,q,r=0}^{\infty} |c_{p,q,r}|^2 \tag{2.7}$$

so that it is a Hilbert space. The star denotes complex conjugation. We remember the reader with the physical interpretation of the state vector. If ψ is the state vector of the particle, then $|\Theta(\psi)_{p,q,r}|^2$ is the probability that the particle be observed in the state $f_{p,q,r}$. If the particle is in the state ψ, then the probability that the particle be observed in the elementary volume $dx\, dy\, dz$ is $|\psi|^2\, dx\, dy\, dz$.

2.3. Functional Spaces

The state vector (resp. the eigenstate decomposition) naturally lies in the space L^2 (resp. l^2). However, in order to prove the forthcoming results, we shall need sharp controls of the state vector and its eigenstate decomposition. It will then appear necessary to consider ψ and $\Theta(\psi)$ as lying in suitable subspaces of L^2 and l^2, which will constitute a convenient framework for our study. We define in the following these spaces.

We first introduce subspaces of the physical space $L^2(\mathbb{R}^3, \mathbb{C})$. The space L^2_α, $\alpha \in \mathbb{N}$, is the space associated with the following norm:

$$\|\psi\|_\alpha^2 = \sum_{\zeta \in \{x, y, z\}} \sum_{\delta_1, \dots, \delta_\alpha \in \{0, 1\}} \|D_{\delta_\alpha, \zeta} \circ \cdots \circ D_{\delta_1, \zeta} \psi\|^2 \tag{2.8}$$

where $\|\cdot\|$ denotes the standard L^2-norm, $D_{0,\zeta}$ is the multiplication operator $D_{0,\zeta}\psi := \zeta\psi$ and $D_{1,\zeta}$ is the derivation operator $D_{1,\zeta}\psi := \partial\psi/\partial\zeta$.

We now introduce the subspaces of l^2 corresponding to the eigenstate representation. For $\alpha \in \mathbb{R}_+$, l^2_α denotes the space of all the sequences c which are decaying so that their weighted norms $\|c\|_\alpha$, $\alpha \in \mathbb{R}_+$ are finite, where:

$$\|c\|_\alpha^2 = \sum_{p,q,r=0}^{\infty} ((1+p)^\alpha + (1+q)^\alpha + (1+r)^\alpha)\, |c_{p,q,r}|^2 \tag{2.9}$$

Both L^2_α and l^2_α are Hilbert and they correspond with each other as stated in the following proposition.

Proposition 2.2. The eigenstate decomposition Θ is a continuous isomorphism from L^2_α onto l^2_α for any $\alpha \in \mathbb{N}$. There exists a constant C_α such that, for any $\psi \in L^2_\alpha$:

$$C_\alpha^{-1} \|\Theta(\psi)\|_\alpha \leqslant \|\psi\|_\alpha \leqslant C_\alpha \|\Theta(\psi)\|_\alpha \tag{2.10}$$

Proof. The proof is given in Appendix A. ∎

2.4. The Harmonic Oscillator Driven by a Uniform Electric Field

Let us assume that the particle possesses a charge q_e. Suppose that we apply an external and homogeneous electric field $\mathscr{E}_0 \mathbf{u}$, where $\mathbf{u}$ is a unit vector of $\mathbb{R}^3$. Let us denote by ε the dimensionless quantity:

$$\varepsilon := \frac{q_e \mathscr{E}_0}{\hbar^{1/2} M^{1/2} \omega^{3/2}} \tag{2.11}$$

It is well-known[14] that the state energies of this system are $(2p + 2q + 2r + 3)/2 - \varepsilon^2/2$ and the corresponding eigenstates $f_p(x - \varepsilon u_x)\ f_q(y - \varepsilon u_y)\ f_r(z - \varepsilon u_z)$ for $p, q, r \in \mathbb{N}$. It means that the spectrum is simply shifted by a constant with respect to the spectrum of the pure harmonic oscillator.

3. EVOLUTION DRIVEN BY A TIME-DEPENDENT ELECTRIC FIELD

3.1. Formulation of the Problem

Let us assume that the particle possesses a charge q_e. Suppose that we apply an external, homogeneous and time-dependent electric field $\mathscr{E}_0 \mathbf{m}(t)$. The dimensionless function $\mathbf{m} = (m_x, m_y, m_z)$ describes the time-fluctuations of the field. This corresponds to an electrostatic potential $\mathscr{E}_0(x m_x(t) + y m_y(t) + z m_z(t))$ and a potential energy $-q_e \mathscr{E}_0(x m_x(t) + y m_y(t) + z m_z(t))$. The dimensionless quantity ε defined by (2.11) is a parameter which characterizes the amplitudes of the fluctuations. The perturbed equation which governs the evolution of the state vector is then:

$$2i \frac{\partial \psi}{\partial t} = -\Delta \psi + (x^2 + y^2 + z^2)\, \psi - 2\varepsilon(x m_x(t) + y m_y(t) + z m_z(t))\, \psi \tag{3.1}$$

Existence and uniqueness of the solution will be stated in Proposition 3.1.

3.2. Scales and Hypotheses

We assume that the amplitudes of the fluctuations are of order $\varepsilon \ll 1$. The $\mathbb{R}^3$-valued function $\mathbf{m}$ is assumed to be zero-mean, stationary and ergodic process under $\mathbb{P}$. We assume that the $\mathbb{R}$-valued random processes m_x, m_y and m_z are independent. We shall denote in the following by $\mathscr{F}_{\tau_0}^{\tau_1}$ the σ-algebra generated by $\sigma(\mathbf{m}(\tau), \tau_0 \leqslant \tau \leqslant \tau_1)$. We shall consider that the process $\mathbf{m}$ is not only ergodic, but also ϕ-mixing, i.e. that there exists a function $t \mapsto \phi(t)$ vanishing as $t \to +\infty$ and belonging to $L^{1/2}(\mathbb{R}_+)$ such that

$$\sup_{\tau > 0} \{ \mathbb{P}(B/A) - \mathbb{P}(B), A \in \mathscr{F}_0^{\tau}, B \in \mathscr{F}_{t+\tau}^{\infty} \} \leqslant \phi(t) \tag{3.2}$$

We introduce also the normalized processes $m_\zeta^\varepsilon(t) = m_\zeta(t/\varepsilon^2)$ and $\psi^\varepsilon(t) = \psi(t/\varepsilon^2)$, and the re-scaled σ-algebra $\mathscr{F}_t^\varepsilon = \mathscr{F}_0^{t/\varepsilon^2}$.

3.3. Statement of the Result

We aim at studying the evolution of the state vector ψ of the particle. The initial state vector at time $t = 0$ is ψ_0, which corresponds to the decomposition $c_0 = \Theta(\psi_0)$. By Proposition 2.2 it is equivalent to study the evolution of its decomposition onto the family of eigenstates $(f_{p,q,r})_{p,q,r \in \mathbb{N}}$, i.e., the corresponding normalized coefficients c^ε:

$$c^\varepsilon_{p,q,r}(t) = \Theta(\psi^\varepsilon(t, \cdot))_{p,q,r} \, e^{i((2p+2q+2r+3)/2)(t/\varepsilon^2)} \tag{3.3}$$

The equation which governs the evolution of c^ε is:

$$\frac{dc^\varepsilon}{dt} = \frac{1}{\varepsilon} m_x\left(\frac{t}{\varepsilon^2}\right) B_x^\varepsilon c^\varepsilon + \frac{1}{\varepsilon} m_y\left(\frac{t}{\varepsilon^2}\right) B_y^\varepsilon c^\varepsilon + \frac{1}{\varepsilon} m_z\left(\frac{t}{\varepsilon^2}\right) B_z^\varepsilon c^\varepsilon \tag{3.4}$$

where the B_ζ^ε are the continuous linear operators from l_α^2 into $l_{\alpha-1}^2$ (for any $\alpha \geqslant 1$) given by:

$$(B_x^\varepsilon c)_{p,q,r} = \frac{i}{\sqrt{2}} (\sqrt{p}\, c_{p-1,q,r} e^{it/\varepsilon^2} + \sqrt{p+1}\, c_{p+1,q,r} e^{-it/\varepsilon^2}) \tag{3.5}$$

and B_y^ε (resp. B_z^ε) acts on the q-index (resp. r-index).

Proposition 3.1. 1. If $c_0 \in l^2_\alpha$ for some $\alpha \geqslant 2$, then for any $\varepsilon > 0$, for almost every realization of m, there exists a unique solution in $\mathbf{C}([0, \infty), l^2_\alpha)$ of Equation (3.4).

2. If the initial state vector $\psi_0 \in L^2_\alpha$ for some $\alpha \geqslant 2$, for any $\varepsilon > 0$, for almost every realization of m, there exists a unique solution ψ in $\mathbf{C}([0, \infty), L^2_\alpha)$ of Equation (3.1) and use have $(\psi^\varepsilon(t, \cdot) = \psi(t/\varepsilon^2, \cdot))$:

$$\psi^\varepsilon(t, x, y, z) = \sum_{p, q, r = 0}^{\infty} c^\varepsilon_{p, q, r}(t)\, e^{-i((2p+2q+2r+3)/2)(t/\varepsilon^2)} f_{p, q, r}(x, y, z) \tag{3.6}$$

Proof. The first point follows from Lemma 5.2. The second point is then a corollary of Proposition 2.2 and basic formulae (A.1) (see Appendix A). ∎

We consider the infinite-dimensional system of linear differential equations starting from $c(0) = c_0$:

$$\begin{aligned} dc = {} & \sqrt{\gamma_x}\, B_{1,x} c\, dW_{1t} + \sqrt{\gamma_x}\, B_{2,x} c\, dW_{2t} + \gamma_x A_x c\, dt \\ & + \sqrt{\gamma_y}\, B_{1,y} c\, dW_{3t} + \sqrt{\gamma_y}\, B_{2,y} c\, dW_{4t} + \gamma_y A_y c\, dt \\ & + \sqrt{\gamma_z}\, B_{1,z} c\, dW_{5t} + \sqrt{\gamma_z}\, B_{2,z} c\, dW_{6t} + \gamma_z A_z c\, dt \end{aligned} \tag{3.7}$$

where W_j, $j = 1,..., 6$ are independent standard Brownian motions, A_ζ and $B_{j,\zeta}$ are the continuous linear operators from l^2_α into $l^2_{\alpha-2}$ $(\alpha \geqslant 2)$ defined by:

$$\begin{aligned} (B_{1,x} c)_{p,q,r} &= -\sqrt{p}\, c_{p-1,q,r} + \sqrt{p+1}\, c_{p+1,q,r} \\ (B_{2,x} c)_{p,q,r} &= i\sqrt{p}\, c_{p-1,q,r} + i\sqrt{p+1}\, c_{p+1,q,r} \\ (A_x c)_{p,q,r} &= -(2p+1)\, c_{p,q,r} \end{aligned} \tag{3.8}$$

$B_{j,y}$ (resp. $B_{j,z}$) acts on the q-index (resp. r-index) and γ_ζ, $\zeta \in \{x, y, z\}$, is given by:

$$\gamma_\zeta = \frac{1}{2} \int_0^\infty \mathbb{E}[m_\zeta(0)\, m_\zeta(t)] \cos(t)\, dt \tag{3.9}$$

γ_ζ is nonnegative because it is proportional to the 1-frequency evaluation of the spectral density function by the Wiener–Khintchine theorem.[15] We can now state our main result.

Theorem 3.2. Let us assume that c_0 belongs to l^2_α for some $\alpha \geqslant 2$.

1. There exists a unique solution c in $\mathbf{C}([0, \infty), l^2_\alpha)$ of Equation (3.7).

2. The processes c^ε converge in distribution in $\mathbf{C}([0, \infty), l^2_\alpha)$ to the continuous Markov process c solution of (3.7) as $\varepsilon \to 0$.

The $|c_{p,q,r}|^2(t)$ given by (3.7) represent the probabilities that the charged particle driven by the random field $\varepsilon\mathbf{m}$ be observed in the state $f_{p,q,r}$ at time t/ε^2 in the limit $\varepsilon \to 0$. Theorem 3.2 is very useful since it allows us to apply the powerful Itô's stochastic calculus to compute all relevant quantities. Before turning to the Section 5 which is devoted to the proof of this theorem, we give remarkable properties of the asymptotic system.

4. SOME PROPERTIES OF THE LIMIT SYSTEM

4.1. Asymptotic Evolution of the State Vector

First we want to underline that the assertion "*the processes ψ^ε converge in* $\mathbf{C}([0, \infty), L^2_\alpha)$" does not hold true, because of the fast varying phases of the eigenstate decomposition of the state vector (3.6). However, the state vector presents the remarkable property that it is time-periodic in the case of the perfect harmonic oscillator. This time period is of duration 4π. As a consequence, if we plot the particle "stroboscopically," i.e., at the regularly spaced times $4\pi k$, $k \in \mathbb{N}$, then the corresponding discontinuous state vector defined by:

$$\tilde{\psi}^\varepsilon(t, x, y, z) = \psi\left(4\pi\left[\frac{t}{4\pi\varepsilon^2}\right], x, y, z\right) \tag{4.1}$$

possesses nice convergence properties in the space of the càd-làg functions $\mathbf{D}$ (the so-called right-continuous with left-limits functions) equipped with the Skorohod topology ($[\tau]$ stands for the integral part of a real number τ).

Proposition 4.1. The processes $\tilde{\psi}^\varepsilon$ converge in distribution in $\mathbf{D}([0, \infty), L^2_\alpha)$ to $\tilde{\psi}$:

$$\tilde{\psi}(t, x, y, z) = \sum_{p,q,r=0}^{\infty} c_{p,q,r}(t)\, f_{p,q,r}(x, y, z) \tag{4.2}$$

where c is described by (3.7). $\tilde{\psi}$ is the unique solution of:

$$d\tilde{\psi} = \sqrt{2\gamma_x}\,\frac{\partial \tilde{\psi}}{\partial x}\,dW_{1t} + i\sqrt{2\gamma_x}\,x\tilde{\psi}\,dW_{2t} - \gamma_x x^2 \tilde{\psi}\,dt + \gamma_x \frac{\partial^2 \tilde{\psi}}{\partial x^2}\,dt$$
$$+ \sqrt{2\gamma_y}\,\frac{\partial \tilde{\psi}}{\partial y}\,dW_{3t} + i\sqrt{2\gamma_y}\,y\tilde{\psi}\,dW_{4t} - \gamma_y y^2 \tilde{\psi}\,dt + \gamma_y \frac{\partial^2 \tilde{\psi}}{\partial y^2}\,dt$$
$$+ \sqrt{2\gamma_z}\,\frac{\partial \tilde{\psi}}{\partial z}\,dW_{5t} + i\sqrt{2\gamma_z}\,z\tilde{\psi}\,dW_{6t} - \gamma_z z^2 \tilde{\psi}\,dt + \gamma_z \frac{\partial^2 \tilde{\psi}}{\partial z^2}\,dt \qquad (4.3)$$

starting from $\tilde{\psi}(0, x, y, z) = \psi_0(x, y, z)$.

More generally, let us fix $\tau \in [0, 4\pi]$. If we plot the particle every times $(\tau + 4\pi k)$, $k \in \mathbb{N}$ and we denote by $\tilde{\psi}^{\varepsilon}_{\tau}$ the corresponding state vectors at these times:

$$\tilde{\psi}^{\varepsilon}_{\tau}(t, x, y, z) = \psi\left(\tau + 4\pi\left[\frac{t}{4\pi\varepsilon^2}\right], x, y, z\right) \qquad (4.4)$$

then the processes $\tilde{\psi}^{\varepsilon}_{\tau}$ converge in distribution in $\mathbf{D}([0, \infty), L^2_{\alpha})$ to $\tilde{\psi}_{\tau}$:

$$\tilde{\psi}_{\tau}(t, x, y, z) = \sum_{p,q,r=0}^{\infty} c_{p,q,r}(t)\, e^{-i((2p+2q+2r+3)/2)\tau} f_{p,q,r}(x, y, z) \qquad (4.5)$$

The convergence at hand is uniform with respect to τ. Indeed the processes $(\tilde{\psi}^{\varepsilon}_{\tau})_{\tau \in [0, 4\pi]}$ actually converge to $(\tilde{\psi}_{\tau})_{\tau \in [0, 4\pi]}$ in $\mathbf{C}([0, 4\pi], \mathbf{D}([0, \infty), L^2_{\alpha}))$. The asymptotic state vector $\tilde{\psi}_{\tau}$ can be derived from $\tilde{\psi}$ through the following equation, in which t is frozen:

$$2i\,\frac{\partial \tilde{\psi}_{\tau}}{\partial \tau} = -\Delta \tilde{\psi}_{\tau} + (x^2 + y^2 + z^2)\,\tilde{\psi}_{\tau}, \qquad \tilde{\psi}_{\tau}(t, x, y, z)|_{\tau=0} = \tilde{\psi}(t, x, y, z) \qquad (4.6)$$

4.2. Conversion of the Fundamental State

We focus here into the following problem. We assume that the state of the particle is the lowest order eigenstate $f_{0,0,0}$ at time $t = 0$. We want to solve the question what is the probability that the particle be in the state $f_{p,q,r}$ at normalized time t.

Proposition 4.2. If $c_{0_{p,q,r}} = \delta_{p0}\,\delta_{q0}\,\delta_{r0}$, then the limit process c satisfies for any $p, q, r \in \mathbb{N}$:

$$\mathbb{E}[\,|c_{p,q,r}|^2\,(t)\,] = \frac{(2\gamma_x t)^p}{(1+2\gamma_x t)^{p+1}} \times \frac{(2\gamma_y t)^q}{(1+2\gamma_y t)^{q+1}} \times \frac{(2\gamma_z t)^r}{(1+2\gamma_z t)^{r+1}} \qquad (4.7)$$

Proof. We denote $C(t) = (\mathbb{E}[\,|c_{p,q,r}|^2\,(t)\,])_{p,q,r \in \mathbb{N}}$. It follows from (5.44) and Itô's formula that C satisfies the homogeneous linear equation starting from $C(0) = \delta_{0p}\,\delta_{0q}\,\delta_{0r}$:

$$\frac{dC}{dt} = 2 \sum_{\zeta \in \{x, y, z\}} \gamma_\zeta H_\zeta C \qquad (4.8)$$

where $(H_x C)_{p,q,r} = pC_{p-1,q,r} - (2p+1)\,C_{p,q,r} + (p+1)\,C_{p+1,q,r}$, and H_y (resp. H_z) acts on the q-index (resp. r-index). Solving this equation provides the result. ▮

Proposition 4.2 shows that the decay of the fundamental state obeys a power law. It means that, starting from the fundamental eigenstate $f_{0,0,0}$, the probability that the particle be in this state at time t decreases at rate t^{-3}.

4.3. Conversion of any Initial State

The result presented in Proposition 4.2 can be generalized to any initial configuration.

Proposition 4.3. If the initial state vector belongs to L_2^2, then the limit process c satisfies for any $p, q, r \in \mathbb{N}$ and for every time t:

$$\mathbb{E}[\,|c_{p,q,r}|^2\,(t)\,] = \sum_{p_0, q_0, r_0 = 0}^{\infty} |c_{0_{p_0,q_0,r_0}}|^2\, \chi_{p_0,p}(2\gamma_x t)\, \chi_{q_0,q}(2\gamma_y t)\, \chi_{r_0,r}(2\gamma_z t) \qquad (4.9)$$

where $\chi_{p_0,p}(t)$ is given by:

$$\chi_{p_0,p}(t) = \sum_{j=(p_0-p)^+}^{p_0} \frac{p_0!\,p!}{(p_0-j)!^2\,j!\,(p-p_0+j)!} \frac{t^{p+2j-p_0}}{(1+t)^{p+p_0+1}} \qquad (4.10)$$

Proof. The proof consists in solving the homogeneous linear system (4.8) with any initial condition. This is just long but straightforward recalculations. ▮

4.4. Energy of the Particle

We have already seen that the processes ψ^ε do not converge globally in $\mathbf{C}([0, \infty), L^2)$. However we can explicitly estimate the variations of an infinity of observables which are constants of motion for the unperturbed Schrödinger equation. We give in the following proposition the first two of them.

Proposition 4.4. Let us denote:

$$M^\varepsilon_{2,x}(t) = \int_{\mathbb{R}^3} \left(\left| \frac{\partial \psi^\varepsilon}{\partial x} \right|^2 + x^2 \, |\psi^\varepsilon|^2 \right) (t, x, y, z) \, dx \, dy \, dz$$

$$M^\varepsilon_{4,x}(t) = \int_{\mathbb{R}^3} \left(\left| \frac{\partial^2 \psi^\varepsilon}{\partial x^2} \right|^2 + \left| x \frac{\partial \psi^\varepsilon}{\partial x} \right|^2 + \left| \frac{\partial x \psi^\varepsilon}{\partial x} \right|^2 + x^4 \, |\psi^\varepsilon|^2 \right) (t, x, y, z) \, dx \, dy \, dz$$

If $\psi_0 \in L^2_2$, then the quantities $M^\varepsilon_{2,x}$ and $M^\varepsilon_{4,x}$ converge in $\mathbf{C}([0, \infty), \mathbb{R}_+)$ to the processes $M_{2,x}$ and $M_{4,x}$ given by:

$$M_{2,x}(t) = M^0_{2,x} + 4\gamma_x^{1/2}(-R^0_{1,x} W_{1t} + I^0_{1,x} W_{2t}) + 2\gamma_x(W_{1t}{}^2 + W_{2t}{}^2) \tag{4.11}$$

$$\begin{aligned} M_{4,x}(t) = {} & M^0_{4,x} + 16\gamma_x^{1/2}(-R^0_{3,x} W_{1t} + I^0_{3,x} W_{2t}) \\ & + 8\gamma_x((M^0_{2,x} + R^0_{2,x}) W_{1t}{}^2 + (M^0_{2,x} - R^0_{2,x}) W_{2t}{}^2 - 2I^0_{2,x} W_{1t} W_{2t}) \\ & + 16\gamma_x^{3/2}(-R^0_{1,x} W_{1t}{}^3 + I^0_{1,x} W_{2t}{}^3 - R^0_{1,x} W_{1t} W_{2t}{}^2 + I^0_{1,x} W_{1t}{}^2 W_{2t}) \\ & + 4\gamma_x^2(W_{1t}{}^2 + W_{2t}{}^2)^2 \end{aligned} \tag{4.12}$$

where the $M^0_{j,x}$'s correspond to ψ_0 and

$$\begin{cases} \begin{pmatrix} R^0_{1,x} \\ I^0_{1,x} \end{pmatrix} = \begin{pmatrix} \mathrm{Re} \\ \mathrm{Im} \end{pmatrix} \left(\sum_{p,q,r} \sqrt{p} \, c^*_{0_{p-1,q,r}} c_{0_{p,q,r}} \right) \\ \begin{pmatrix} R^0_{2,x} \\ I^0_{2,x} \end{pmatrix} = \begin{pmatrix} \mathrm{Re} \\ \mathrm{Im} \end{pmatrix} \left(\sum_{p,q,r} \sqrt{p(p+1)} \, c^*_{0_{p-1,q,r}} c_{0_{p+1,q,r}} \right) \\ \begin{pmatrix} R^0_{3,x} \\ I^0_{3,x} \end{pmatrix} = \begin{pmatrix} \mathrm{Re} \\ \mathrm{Im} \end{pmatrix} \left(\sum_{p,q,r} \sqrt{p^3} \, c^*_{0_{p-1,q,r}} c_{0_{p,q,r}} \right) \end{cases} \tag{4.13}$$

As a consequence the mean values of $M_{j,x}$ obey:

$$\begin{aligned} \mathbb{E}[M_{2,x}(t)] &= M^0_{2,x} + 4\gamma_x t \\ \mathbb{E}[M_{4,x}(t)] &= M^0_{4,x} + 16 M^0_{2,x} \gamma_x t + 32\gamma_x^2 t^2 \end{aligned} \tag{4.14}$$

Similar results hold true for $M_{j,y}$ and $M_{j,z}$.

Proof. Expressing $M^{\varepsilon}_{j,x}$ in terms of the coefficients c^{ε}, we get from Theorem 3.2 that $M^{\varepsilon}_{2,x}$ and $M^{\varepsilon}_{4,x}$ converge to the processes $M_{2,x}$ and $M_{4,x}$ given by:

$$\begin{aligned} M_{2,x}(t) &= \sum_{p,q,r} (2p+1)\, |c_{p,q,r}|^2(t) \\ M_{4,x}(t) &= \sum_{p,q,r} (4p^2+4p+3)\, |c_{p,q,r}|^2(t) \end{aligned} \tag{4.15}$$

where c is given by (3.7). Applying Itô's formula and standard formulae of stochastic calculus we get the representations of $M_{j,x}$ in terms of the Brownian motions. ∎

The expressions of the above Proposition can be simplified in the case where the initial state vector is some eigenstate f_{p_0,q_0,r_0}.

Corollary 4.5. With the same notations as in Proposition 4.4, if $c_{0_{p,q,r}} = \delta_{pp_0}\delta_{qq_0}\delta_{rr_0}$, then:

$$M_{2,x}(t) = (2p_0+1) + 2\gamma_x({W_{1t}}^2 + {W_{2t}}^2) \tag{4.16}$$

$$M_{4,x}(t) = (4p_0^2+4p_0+3) + 8\gamma_x(2p_0+1)({W_{1t}}^2 + {W_{2t}}^2) + 4\gamma_x^2({W_{1t}}^2 + {W_{2t}}^2)^2 \tag{4.17}$$

The linear growth of the averaged energy $\mathbb{E}[M_{2,\zeta}(t)]$ was also established in ref. 10. In this paper[10] the authors considered a one-dimensional harmonic oscillator and conjectured that "the energy behaves like the square of a Gaussian random variable with variance proportional to time when the random potential $m(t)$ has a positive Lyapunov exponent." They proved the conjecture for an explicit example. From (4.11, 4.16) we can claim that, whenever $m(t)$ is ϕ-mixing with $\phi \in L^{1/2}(\mathbb{R}_+)$, the energy regarded as a time-process behaves like the sum of the squares of two independent and identically distributed Brownian motions. As a consequence, if $R^0_{1,x} = I^0_{1,x} = 0$, then at some fixed time t the energy $M_{2,x}(t)$ obeys an exponential distribution with density:

$$p_{M_{2,x}}(E) = \frac{1}{4\gamma_x t} \exp\left(-\frac{E - M^0_{2,x}}{4\gamma_x t} \right) \mathbb{1}_{E \geq M^0_{2,x}} \tag{4.18}$$

Furthermore, if $R^0_{1,\zeta} = I^0_{1,\zeta} = 0$ for $\zeta = x, y, z$, then the total energy $M_2 := M_{2,x} + M_{2,y} + M_{2,z}$ for the three-dimensional harmonic oscillator behaves like the sum of the squares of six independent Brownian motions.

If $\gamma_x = \gamma_y = \gamma_z = \gamma$, then at some fixed time t the total energy $M_2(t)$ obeys a Gamma distribution with density:

$$p_{M_2}(E) = \frac{1}{2} \frac{(E - M_2^0)^2}{(12\gamma t)^3} \exp\left(-\frac{E - M_2^0}{12\gamma t} \right) \mathbb{1}_{E \geqslant M_2^0} \tag{4.19}$$

4.5. Interpretation of the Transition Probabilities in Terms of a Jump Process

The transition probabilities $C_{p,q,r}(t) := \mathbb{E}[|c_{p,q,r}|^2(t)]$ satisfy the system (4.8) starting from the initial configuration $C_{p,q,r}(0) = |c_{0_{p,q,r}}|^2$. We shall show that they can be regarded as the statistical distribution of a jump process. We denote $\gamma = (\gamma_x, \gamma_y, \gamma_z)$, $\mathbf{e}^x = (1, 0, 0)$, $\mathbf{e}^y = (0, 1, 0)$, $\mathbf{e}^z = (0, 0, 1)$, and $\mathbf{u} = \mathbf{e}^x + \mathbf{e}^y + \mathbf{e}^z = (1, 1, 1)$. Let $N(t) = (N_x(t), N_y(t), N_z(t))$ be the Markov process with state space $E = \mathbb{N}^3$ and infinitesimal generator $\mathscr{L}$:

$$\mathscr{L} = 2 \sum_{\zeta \in \{x, y, z\}} \gamma_\zeta N_\zeta \nabla_\zeta^- + \gamma_\zeta (N_\zeta + 1) \nabla_\zeta^+ \tag{4.20}$$

where $\nabla_\zeta^- g(N) = g(N - \mathbf{e}^\zeta) - g(N)$ and $\nabla_\zeta^+ g(N) = g(N + \mathbf{e}^\zeta) - g(N)$. $(N(t))_{t \geqslant 0}$ is a time-homogeneous jump process defined on some probability space $(\Omega, \mathscr{F}, \mathbf{P})$. It is a birth-and-death process of three distinct and independent populations with birth rates $\lambda_n^\zeta = 2\gamma_\zeta(n+1)$ and death rates $\mu_n^\zeta = 2\gamma_\zeta n$. It means that the population levels change only through transitions to their nearest neighbors. If at time t the process is in state $n = (n_x, n_y, n_z)$, then the probability that between t and $t + h$ the transition $n \mapsto (n_x + 1, n_y, n_z)$ occurs equals $\lambda_n^x h + o(h)$, and the probability of $n \mapsto (n_x - 1, n_y, n_z)$ equals $\mu_n^x h + o(h)$. The same holds for the other indices. The probability that during $(t, t+h)$ more than one change occurs is $o(h)$. It can then be checked that $C_{p,q,r}(t) = \mathbf{P}(N_x(t) = p, N_y(t) = q, N_z(t) = r)$, where $\mathbf{P}$ stands for the distribution of the paths $(N(t))_{t \geqslant 0}$ starting at time $t = 0$ with the initial distribution $\mathbf{P}(N_x(0) = p,\ N_y(0) = q,\ N_z(0) = r) = C_{p,q,r}(0)$. This interpretation of the transition probabilities may help to solve problems. Let us examine one of them. The explicit results of Subsection 4.2. show that the decrease rate of the fundamental mode is $\sim t^{-3}$. We shall prove here that this is a very general feature. In the following we assume that $\gamma_\zeta \neq 0$ and consequently $\gamma_\zeta > 0$ for $\zeta = x, y, z$.

Proposition 4.6. We denote:

$$P_t^\delta(a_x, a_y, a_z) := \delta^{-3} \mathbb{E}\left[|c_{[a_x/\delta], [a_y/\delta], [a_z/\delta]}|^2 \left(\frac{t}{\delta} \right) \right] \tag{4.21}$$

which is proportional to the probability that the particle be observed in the eigenstate $f_{[a_x/\delta],[a_y/\delta],[a_z/\delta]}$ at time t/δ in the asymptotic regime (c is the solution of (3.7)).

If the initial state vector belongs to L^2_2, then P^δ_t converge as a function of $\mathbf{C}((0,\infty), L^1(\mathbb{R}^3_+))$ to the continuous function P_t given by:

$$P_t(a_x, a_y, a_z) = \frac{1}{8\gamma_x\gamma_y\gamma_z t^3}\, e^{-(a_x/2\gamma_x t)-(a_y/2\gamma_y t)-(a_z/2\gamma_z t)} \tag{4.22}$$

This proposition establishes that the probability that the particle be observed in some given eigenstate decreases as t^{-3}.

Proof. This proposition can be established directly by studying the exact formulae of Proposition 4.3. However we shall give a proof which is independent from these results and which is based upon the above interpretation of the limit system in terms of the jump process $N(t)$.

Step 1. The re-scaled process. Let us denote by N^δ the re-scaled $\mathscr{F}^\delta_t := \mathscr{F}^{t/\delta}_0$-adapted process defined by:

$$N^\delta(t) = \delta N(t/\delta) \tag{4.23}$$

N^δ is a Markov process with infinitesimal generator $\mathscr{L}^\delta$:

$$\begin{aligned}\mathscr{L}^\delta g(\mathbf{a}) = 2\delta^{-2} \sum_{\zeta\in\{x,y,z\}} & \gamma_\zeta(a_\zeta(g(\mathbf{a}-\delta\mathbf{e}^\zeta)-g(\mathbf{a}))\\ & + (a_\zeta+\delta)(g(\mathbf{a}+\delta\mathbf{e}^\zeta)-g(\mathbf{a})))\end{aligned} \tag{4.24}$$

where $\mathbf{a} = (a_x, a_y, a_z) \in \mathbb{R}^3_+$. For any $g \in \mathbf{C}^3(\mathbb{R}^3_+, \mathbb{R})$ with bounded derivatives, the process

$$M^\delta_g(t) := g(N^\delta(t)) - g(\delta N(0)) - \int_0^t \mathscr{L}^\delta g(N^\delta(u))\, du \tag{4.25}$$

is a $\mathscr{F}^\delta_t$-martingale. Applying this statement with $g_\zeta(\mathbf{a}) = a_\zeta$ for $\zeta = x$, y and z yields:

$$\mathbb{E}[N^\delta(t+s)/\mathscr{F}^\delta_t] = N^\delta(t) + 2\gamma s \tag{4.26}$$

Besides, by Doob's inequality:

$$\mathbf{E}[\sup_{u \in [0,s]} (M_g^\delta(t+u) - M_g^\delta(t))^2 / \mathscr{F}_t^\delta]$$
$$\leqslant 4 \sup_{u \in [0,s]} \mathbf{E}[M_g^\delta(t+u)^2 - M_g^\delta(t)^2 / \mathscr{F}_t^\delta]$$
$$= 4\mathbf{E}\left[\int_t^{t+s} (\mathscr{L}^\delta g^2 - 2g\mathscr{L}^\delta g)(N^\delta(u))\, du / \mathscr{F}_t^\delta\right] \tag{4.27}$$

Taking $g_\zeta(\mathbf{a}) = a_\zeta$, this establishes in particular that:

$$\mathbf{E}[\sup_{u \in [0,s]} |N^\delta(t+u) - N^\delta(t) - 2\gamma u|^2 / \mathscr{F}_t^\delta]$$
$$\leqslant 16 N^\delta(t) \cdot \gamma s + 16\gamma \cdot \gamma s^2 + 8\gamma \cdot \mathbf{u}\, \delta s \tag{4.28}$$

Step 2. Tightness. In order to prove the tightness of the process N^δ in $\mathbf{D}([0, \infty), \mathbb{R}^3)$, we use the Aldous criteria (see Lemma 5.6) which is fulfilled in view of (4.26) and (4.28).

Step 3. Convergence. Expanding the right-hand side of (4.24), we get that there exists a constant K such that:

$$|\mathscr{L}^\delta g(\mathbf{a}) - \bar{\mathscr{L}} g(\mathbf{a})| \leqslant K(1 + \|g^{(3)}\|_\infty |\mathbf{a}|)\, \delta \tag{4.29}$$

where $\bar{\mathscr{L}}$ is the generator defined by:

$$\bar{\mathscr{L}} = 2 \sum_{\zeta \in \{x, y, z\}} \gamma_\zeta \frac{\partial}{\partial a_\zeta} a_\zeta \frac{\partial}{\partial a_\zeta} \tag{4.30}$$

this yields that for any functions $g, h_1, \ldots, h_n \in \mathbf{C}_b^3$ and for any $0 \leqslant \tau_1 < \cdots < \tau_n \leqslant t_0 \leqslant t_1$:

$$\lim_{\delta \to 0} \mathbf{E}\left[h_1(N^\delta(\tau_1)) \cdots h_n(N^\delta(\tau_n)) \left(g(N^\delta(t_1)) - g(N^\delta(t_0)) - \int_{t_0}^{t_1} \bar{\mathscr{L}} g(N^\delta(s))\, ds \right)\right] = 0 \tag{4.31}$$

If we consider a sub-sequence δ_p such that the processes N^{δ_p} converge in distribution to some limit $\bar{N}$ as $p \to \infty$, then we get:

$$\mathbf{E}\left[h_1(\bar{N}(\tau_1)) \cdots h_n(\bar{N}(\tau_n)) \left(g(\bar{N}(t_1)) - g(\bar{N}(t_0)) - \int_{t_0}^{t_1} \bar{\mathscr{L}} g(\bar{N}(s))\, ds \right)\right] = 0 \tag{4.32}$$

which means that $\bar{N}$ is solution of the martingale problem associated with the generator $\bar{\mathscr{L}}$. This problem is well-posed and admits a unique solution, which is the distribution of a Markov diffusion process with infinitesimal generator $\bar{\mathscr{L}}$ and continuous paths in $\mathbf{C}([0, \infty), \mathbb{R}^3)$. This consequently yields the convergence in distribution of N^δ to $\bar{N}$ as $\delta \to 0$.

If the initial decomposition c_0 belongs to l_2^2, then N^δ starts from the distribution $\mathbf{P}(N^\delta(0) = \delta p, \delta q, \delta r) = |c_{0_{p,q,r}}|^2$, and consequently, in the limit $\delta \to 0$, $\bar{N}$ starts from the initial distribution $\mathbf{P}(\bar{N}(0) = (0, 0, 0)) = 1$. Applying the generator $\bar{\mathscr{L}}$, we then find that, for any $(n_x, n_y, n_z) \in \mathbb{R}^3$:

$$\mathbf{E}[\bar{N}_x(t)^{n_x} \bar{N}_y(t)^{n_y} \bar{N}_z(t)^{n_z}] = n_x!\, n_y!\, n_z!\, (2\gamma_x t)^{n_x} (2\gamma_y t)^{n_y} (2\gamma_z t)^{n_z} \tag{4.33}$$

It proves that the measures $P_t^\delta\, da_x\, da_y\, da_z$ weakly converge to $P_t\, da_x\, da_y\, da_z$ as $\delta \to 0$. With some more work we can show the convergence as stated in the Proposition. ▮

4.6. Position of the Particle

We can also give some piece of information about the position of the particle. Since the state vector periodically oscillates, we cannot deal with the instantaneous position, but we can efficiently work on the locally time-averaged position of the particle.

Proposition 4.7. Let us assume that $c_0 \in l_2^2$. For any $n \leqslant 4$, we denote by $\overline{x^n}_\tau^\varepsilon$ the mean nth power of the position of the particle in the x-direction at time $\tau + 4\pi[t/4\pi\varepsilon^2]$:

$$\overline{x^n}_\tau^\varepsilon(t) := \int_{\mathbb{R}^3} x^n\, |\tilde{\psi}_\tau^\varepsilon|^2\, (t, x, y, z)\, dx\, dy\, dz \tag{4.34}$$

Let $\overline{\overline{x^n}}^\varepsilon$ be the mean nth power of the position of the particle time-averaged over a local period 4π:

$$\overline{\overline{x^n}}^\varepsilon(t) := \frac{1}{4\pi} \int_0^{4\pi} \overline{x^n}_\tau^\varepsilon(t)\, d\tau \tag{4.35}$$

Then the processes $\overline{\overline{x^n}}^\varepsilon$ converge in distribution in $\mathbf{C}([0, \infty), \mathbb{R})$ to the processes $\overline{\overline{x^n}}$ defined by:

$$\overline{\overline{x^n}}(t) = \begin{cases} 0 & \text{if} \quad n = 1, 3 \\ 2^{-n/2} M_{n,x}(t) & \text{if} \quad n = 2, 4 \end{cases} \tag{4.36}$$

where the $M_{n,x}(t)$'s are given by (4.11, 4.12).

Proof. We shall prove a more general result. Let us assume that $c_0 \in l_\alpha^2$ for some $\alpha \geqslant 2$. Let g be a smooth function of $\mathbf{C}(\mathbb{R}^3, \mathbb{R})$ with polynomial growth of degree at most 2α. We can study the asymptotic behavior of the following process, function of the position of particle:

$$G^\varepsilon(t) := \frac{1}{4\pi} \int_{\tau=0}^{4\pi} \int_{\mathbb{R}^3} g(x, y, z) \, |\tilde{\psi}_\tau^\varepsilon|^2 (t, x, y, z) \, dx \, dy \, dz \, d\tau \tag{4.37}$$

We find from the results of Subsection 4.1. that the processes G^ε converge in distribution in $\mathbf{C}([0, \infty), \mathbb{R})$ to G defined by:

$$G(t) := \frac{1}{4\pi} \int_{\tau=0}^{4\pi} \int_{\mathbb{R}^3} g(x, y, z) \, |\tilde{\psi}_\tau|^2 (t, x, y, z) \, dx \, dy \, dz \, d\tau \tag{4.38}$$

where $\tilde{\psi}_\tau$ is defined by (4.6) and (4.3). If we choose $g(x, y, z) = x^n$, then we get that $\overline{\overline{x^n}}^\varepsilon$ converges in $\mathbf{C}([0, \infty), \mathbb{R})$ to the process $\overline{\overline{x^n}}$ defined by:

$$\overline{\overline{x^n}}(t) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 2^{-N} \sum\limits_{p, q, r} \beta_N(p, q, r) \, |c_{p, q, r}|^2 (t), & \text{if } n = 2N \text{ is even} \end{cases} \tag{4.39}$$

where c is given by (3.7) and β_N is defined recursively by:

$$\beta_{N+1}(p, q, r) = (p+1) \, \beta_N(p+1, q, r) + p \beta_N(p-1, q, r), \text{ and } \beta_0 \equiv 1 \tag{4.40}$$

The result then follows obviously from Proposition 4.4. ∎

5. PROOF OF THE MAIN THEOREM

This section is devoted to the proof of Theorem 3.2. For the sake of simplicity in the notations we establish the result in the case of space dimension 1. We can easily extend the proof to any finite dimension, so that the result as stated in Theorem 3.2, which holds in space dimension 3, is actually a straightforward generalization of the one-dimensional case. In view of the following lemma,(16) it is sufficient to show that the processes c^ε converge in distribution in $\mathbf{D}([0, \infty), l_\alpha^2)$.

Lemma 5.1. Let (E, d) be a metric space and $T > 0$. If f_n, $n \geqslant 1$ and f are functions in $\mathbf{C}([0, T], E)$ such that the sequence f_n converges in $\mathbf{D}([0, T], E)$ to f, then the sequence f_n actually converges in $\mathbf{C}([0, T], E)$ to f. ∎

5.1. The ϵ-Process

We introduce a sequence of auxiliary processes $c^{\varepsilon, N} \in \mathbf{D}([0, \infty), l^2_\alpha)$, which are finite-dimensional approximations of c^ε, the $c^{\varepsilon, N}$ taking their values in the finite-dimensional subspace H_N of l^2_α:

$$H_N = \{c \in l^2_\alpha,\ c_p = 0 \text{ if } p > N\} \tag{5.1}$$

Defining the projection Π_N from l^2_α onto H_N by $(\Pi_N c)_p = c_p$ if $0 \leqslant p \leqslant N$ and 0 otherwise, we denote by $c^{\varepsilon, N}$ the unique solution in $\mathbf{C}([0, \infty), H_N)$ of:

$$\frac{dc^{\varepsilon, N}}{dt} = \frac{1}{\varepsilon} m^\varepsilon(t)\, \Pi_N \circ B^\varepsilon c^{\varepsilon, N}, \qquad c^{\varepsilon, N}(0) = \Pi_N(c_0) \tag{5.2}$$

where B^ε is given by (3.5) (we drop the index x from the notations). Existence and uniqueness are obvious since the evolution of $c^{\varepsilon, N}$ is actually governed by a finite-dimensional system of linear differential equations. Indeed it can be readily checked that $c^{\varepsilon, N}_p \equiv \tilde{c}^{\varepsilon, N}_p$ for $0 \leqslant p \leqslant N$, where $\tilde{c}^{\varepsilon, N} \in \mathbf{D}([0, \infty), \mathbb{C}^{N+1})$ is governed by the linear system:

$$\frac{d\tilde{c}^{\varepsilon, N}}{dt} = \frac{1}{\varepsilon} m^\varepsilon(t)\, \mathbf{F}^{\varepsilon, N} \tilde{c}^{\varepsilon, N}, \qquad \tilde{c}^{\varepsilon, N}_p(0) = c_{0_p} \qquad \text{for} \quad 0 \leqslant p \leqslant N \tag{5.3}$$

$\mathbf{F}^{\varepsilon, N}$ is the linear mapping from $\mathbb{C}^{N+1}$ into $\mathbb{C}^{N+1}$ given by: $\mathbf{F}^{\varepsilon, N} = \Pi_N \circ B^\varepsilon \circ \Pi_N^{-1}$ (Π_N^{-1} maps $\mathbb{C}^{N+1}$ into l^2_α by completing any finite-dimensional vector with zeros).

Lemma 5.2. Assume that $c_0 \in l^2_\alpha$ for some $\alpha \in \mathbb{N}$, $\alpha \geqslant 1$.

1. There exists a unique solution c^ε in $\mathbf{C}([0, \infty), l^2_\alpha)$ of (3.4) almost surely. Moreover, for any $t \geqslant 0$, there exists a constant $K_{\alpha, t}$ such that:

$$\sup_{\varepsilon \in (0, 1)} \mathbb{E}[\|c^\varepsilon(t)\|^2_\alpha] \leqslant K_{\alpha, t} \tag{5.4}$$

2. For any $\delta > 0$, $t > 0$, there exists a function $K_{\delta, \alpha, t}(N)$ which goes to 0 as $N \to \infty$ such that:

$$\sup_{\varepsilon \in (0, 1)} \mathbb{P}\Big(\sup_{s \in [0, t]} \|c^{\varepsilon, N}(s) - c^\varepsilon(s)\|_\alpha \geqslant \delta \Big) \leqslant K_{\delta, \alpha, t}(N) \tag{5.5}$$

3. $\|c^\varepsilon(t)\| = \|c_0\|$ for any $t \geqslant 0$ almost surely.

Proof. We first introduce some notations. If $(c_p)_{p \in \mathbb{N}}$ is a sequence of complex numbers, then we define for any $p \in \mathbb{N}$:

$$
\begin{aligned}
(J_1 c)_p &= \sqrt{2}\,(-\sqrt{p}\,\mathrm{Im}(c_{p-1}c_p^*) + \sqrt{p+1}\,\mathrm{Im}(c_p c_{p+1}^*)) \\
(J_2 c)_p &= \sqrt{2}\,(\sqrt{p}\,\mathrm{Re}(c_{p-1}c_p^*) - \sqrt{p+1}\,\mathrm{Re}(c_{p+1} c_p^*)) \\
H_{1t}^\varepsilon c &= (h_1 c)\,\xi_1^\varepsilon(t) + \mathrm{Re}(h_2 c)\,\xi_1^\varepsilon(t) + \mathrm{Im}(h_2 c)\,\xi_2^\varepsilon(t) \\
H_{2t}^\varepsilon c &= (h_1 c)\,\xi_2^\varepsilon(t) - \mathrm{Re}(h_2 c)\,\xi_2^\varepsilon(t) + \mathrm{Im}(h_2 c)\,\xi_1^\varepsilon(t) \\
(h_1 c)_p &= (p+1)\,|c_{p+1}|^2 + p\,|c_{p-1}|^2 - (2p+1)\,|c_p|^2 \\
(h_2 c)_p &= 2\sqrt{p(p+1)}\,c_{p-1}c_{p+1}^* - \sqrt{p(p-1)}\,c_{p-2}c_p^* \\
&\quad - \sqrt{(p+1)(p+2)}\,c_p c_{p+2}^* \\
\xi_j^\varepsilon(t) &= \begin{cases} \cos(t/\varepsilon^2) & \text{for } j=1 \\ -\sin(t/\varepsilon^2) & \text{for } j=2 \end{cases}
\end{aligned}
\tag{5.6}
$$

By convention $c_{-2} = c_{-1} = 0$. The time-derivatives of $|c_p^{\varepsilon,N}|^2$ can be expressed in terms of J_j and H_j^ε. On the one hand, for $p = 0,..., N$,

$$
\frac{d\,|c_p^{\varepsilon,N}|^2(t)}{dt} = \frac{1}{\varepsilon} m^\varepsilon(t) \sum_{j=1}^{2} \xi_j^\varepsilon(t)(J_j c^{\varepsilon,N})_p(t) \tag{5.7}
$$

On the other hand, for $p = 0,..., N-1$, for $j = 1, 2$:

$$
\frac{d(J_j c^{\varepsilon,N})_p(t)}{dt} = \frac{1}{\varepsilon} m^\varepsilon(t)(H_j^\varepsilon c^{\varepsilon,N})_p(t) \tag{5.8}
$$

For $p = N$, for $j = 1, 2$, we have:

$$
\frac{d(J_j c^{\varepsilon,N})_p(t)}{dt} = \frac{1}{\varepsilon} m^\varepsilon(t)(H_j^\varepsilon c^{\varepsilon,N})_p(t) + \frac{1}{\varepsilon} m^\varepsilon(t)\,\xi_j^\varepsilon(t)(N+1)\,|c_N^{\varepsilon,N}|^2(t) \tag{5.9}
$$

Besides, since $(1+p)^k - p^k \leqslant k(1+p)^{k-1}$, straightforward calculations establish that, for any $c \in l_{\alpha+1}^2$, for $j = 1, 2$:

$$
\left| \sum_{p=0}^{\infty} (1+p)^\alpha (J_j c)_p \right| \leqslant \alpha 2^\alpha \|c\|_{\alpha-1/2}^2 \tag{5.10}
$$

$$
\left| \sum_{p=0}^{\infty} (1+p)^\alpha (h_1 c)_p \right| \leqslant 3\alpha^2 2^{\alpha-1} \|c\|_{\alpha-1}^2 \tag{5.11}
$$

$$
\left| \sum_{p=0}^{\infty} (1+p)^\alpha (h_2 c)_p \right| \leqslant \alpha^2 3^\alpha \|c\|_{\alpha-1}^2 \tag{5.12}
$$

If $c \in H^N$, then we can substitute the partial sum $\sum_{p=0}^{N}$ for the infinite sum in the left-hand side of (5.10) and (5.12) for J_1, J_2 and h_2. However we have to take care that in the case of h_1 we have:

$$\sum_{p=0}^{\infty} (1+p)^\alpha (h_1 c)_p = \sum_{p=0}^{N} (1+p)^\alpha (h_1 c)_p + (N+1)(N+2)^\alpha |c_N|^2 \tag{5.13}$$

We remember the reader with the techniques developed by Kurtz[17] and Kushner.[18] Let $T_0 > 0$ be fixed. $\mathscr{M}^\varepsilon$ denotes the set of all $\mathscr{F}^\varepsilon$-measurable functions $f(t)$ for which $\sup_{t \leqslant T_0} \mathbb{E}[|f(t)|] < \infty$. Following Kurtz,[17] define the limit $p-lim$ and the operator $\mathscr{A}^\varepsilon$ as follows. Let $f(\cdot)$ and $f^\delta(\cdot)$ be in $\mathscr{M}^\varepsilon$ for each $\delta > 0$. Then we say that $f = p - \lim_\delta f^\delta$ if and only if:

$$\sup_{t,\,\delta} \mathbb{E}[|f^\delta(t)|] < \infty, \qquad \lim_{\delta \to 0} \mathbb{E}[|f^\delta(t) - f(t)|] = 0 \text{ for each } t \tag{5.14}$$

We say that $f(\cdot) \in \mathscr{D}(\mathscr{A}^\varepsilon)$, the domain of $\mathscr{A}^\varepsilon$, and $\mathscr{A}^\varepsilon f = g$ if $f(\cdot)$ and $g(\cdot)$ are in $\mathscr{M}^\varepsilon$ and

$$p - \lim_{\delta \to 0} \left(\frac{\mathbb{E}[f(t+\delta)/\mathscr{F}_t^\varepsilon] - f(t)}{\delta} - g(t) \right) = 0 \tag{5.15}$$

For purposes of the sequel, the most useful property of $\mathscr{A}^\varepsilon$ is given by the following Proposition:[17]

Proposition 5.3. Let $f(\cdot) \in \mathscr{D}(\mathscr{A}^\varepsilon)$. Then $f(t) - \int_0^t \mathscr{A}^\varepsilon f(u)\, du$ is a $\mathscr{F}^\varepsilon$-martingale.

We denote by f_j^ε the $\mathscr{F}^\varepsilon$-adapted functions:

$$f_1^\varepsilon(t) = \int_{t/\varepsilon^2}^{\infty} \cos(s)\, \mathbb{E}[m(s)/\mathscr{F}_t^\varepsilon]\, ds, \qquad f_2^\varepsilon(t) = -\int_{t/\varepsilon^2}^{\infty} \sin(s)\, \mathbb{E}[m(s)/\mathscr{F}_t^\varepsilon]\, ds \tag{5.16}$$

Since m is ϕ-mixing, $f_j^\varepsilon(t) \in L^\infty(\Omega)$, i.e. $\|f_j^\varepsilon(t)\|_\infty \leqslant f_\infty$, where:

$$f_\infty = \|m\|_\infty \int_0^\infty \phi(s)\, ds \tag{5.17}$$

Furthermore $f_j^\varepsilon \in \mathscr{D}(\mathscr{A}^\varepsilon)$ and

$$\mathscr{A}^\varepsilon f_j^\varepsilon(t) = -\varepsilon^{-2} m^\varepsilon(t)\, \xi_j^\varepsilon(t) \tag{5.18}$$

Step 1: For any $t \geqslant 0$, there exists a constant $K_{\alpha,t}$ such that, for any family of $\mathscr{F}^{\varepsilon}$-adapted stopping times T^{ε}:

$$\sup_{\varepsilon \in (0,1)} \sup_{N \in \mathbb{N}} \sup_{s \in [0,t]} \mathbb{E}[\|c^{\varepsilon,N}(s \wedge T^{\varepsilon})\|_{\alpha}^{2}] \leqslant K_{\alpha,t} \tag{5.19}$$

Let T^{ε} be a family of $\mathscr{F}^{\varepsilon}$-adapted stopping times. We aim at studying the process $|c_p^{\varepsilon,N}|^2$, whose time-derivative (5.7) exhibits a $O(\varepsilon^{-1})$-term. We apply the perturbed test function method (see ref. 18, Section 6) in order to get rid off this $O(\varepsilon^{-1})$-term. We therefore consider the function $|c_p^{\varepsilon,N}|^2 + \varepsilon \sum_{j=1}^{2} (J_j c^{\varepsilon,N})_p f_j^{\varepsilon}$, and we get from Proposition 5.3 and (5.8, 5.18) that, for any $p = 0,..., N-1$:

$$\begin{aligned} M_p^{\varepsilon,N}(t) :=& (|c_p^{\varepsilon,N}|^2(t) - |c_{0_p}^{N}|^2) + \varepsilon \sum_{j=1}^{2} ((J_j c^{\varepsilon,N})_p(t) f_j^{\varepsilon}(t) - (J_j c_0^N)_p f_j^{\varepsilon}(0)) \\ & - \sum_{j=1}^{2} \int_0^t (H_j^{\varepsilon} c^{\varepsilon,N})_p(t') m^{\varepsilon}(t') f_j^{\varepsilon}(t')\, dt' \end{aligned} \tag{5.20}$$

is a $\mathscr{F}^{\varepsilon}$-martingale. For $p = N$, we must add some correction in order to take into account (5.9) so that:

$$M_N^{\varepsilon,N}(t) - (N+1) \sum_{j=1}^{2} \int_0^t |c_N^{\varepsilon,N}|^2(t') m^{\varepsilon}(t') \xi_j^{\varepsilon}(t') f_j^{\varepsilon}(t')\, dt' \tag{5.21}$$

is a $\mathscr{F}^{\varepsilon}$-martingale. Expanding $\|c^{\varepsilon,N}\|_{\alpha}^2$ in terms of $M_p^{\varepsilon,N}$, $p = 0,..., N$, we then get by taking into account (5.13):

$$\begin{aligned} &\mathbb{E}[\|c^{\varepsilon,N}\|_{\alpha}^{2}(t \wedge T^{\varepsilon})] \\ &\quad \leqslant \|c_0^N\|_{\alpha}^2 + \varepsilon f_{\infty} \sum_{j=1}^{2} \left| \sum_{p=0}^{\infty} (1+p)^{\alpha} (J_j c_0^N)_p \right| \\ &\quad + \varepsilon f_{\infty} \sum_{j=1}^{2} \mathbb{E}\left[\left| \sum_{p=0}^{\infty} (1+p)^{\alpha} (J_j c^{\varepsilon,N})_p (t \wedge T^{\varepsilon}) \right| \right] \\ &\quad + C_0 \sum_{j=1}^{2} 2^j \int_0^t \mathbb{E}\left[\left| \sum_{p=0}^{\infty} (1+p)^{\alpha} (h_j c^{\varepsilon,N})_p (t' \wedge T^{\varepsilon}) \right| \right] dt' \\ &\quad + 2C_0\, |(N+1)(N+2)^{\alpha} - (N+1)^{\alpha+1}| \int_0^t \mathbb{E}[|c_N^{\varepsilon,N}|^2(t' \wedge T^{\varepsilon})]\, dt' \end{aligned} \tag{5.22}$$

where $C_0 = \|m\|_\infty f_\infty$. Applying the estimates (5.10), (5.11) and (5.12):

$$\begin{aligned}
&\mathbb{E}[\|c^{\varepsilon,N}\|_\alpha^2(t \wedge T^\varepsilon)] \\
&\quad \leqslant \|c_0^N\|_\alpha^2 + 2^{\alpha+1}\alpha\varepsilon f_\infty(\|c_0^N\|_{\alpha-1/2}^2 + \mathbb{E}[\|c^{\varepsilon,N}(t \wedge T^\varepsilon)\|_{\alpha-1/2}^2]) \\
&\qquad + C_\alpha \int_0^t \mathbb{E}[\|c^{\varepsilon,N}\|_{\alpha-1}^2(t' \wedge T^\varepsilon)]\, dt' \\
&\qquad + 2\alpha(N+2)^\alpha C_0 \int_0^t \mathbb{E}[|c_N^{\varepsilon,N}|^2(t' \wedge T^\varepsilon)]\, dt' \qquad (5.23)
\end{aligned}$$

where $C_\alpha = (3\alpha^2 2^\alpha + 4\alpha^2 3^\alpha)\, C_0$. Denoting $K_\alpha = C_\alpha + \alpha 2^{\alpha+1} C_0$, and replacing $\|c^{\varepsilon,N}\|_\alpha^2$ for $\|c^{\varepsilon,N}\|_{\alpha-1}^2$ and $\|c^{\varepsilon,N}\|_{\alpha-1/2}^2$ in the right-hand member, this inequality can be simplified:

$$\begin{aligned}
\mathbb{E}[\|c^{\varepsilon,N}(t \wedge T^\alpha)\|_\alpha^2] \leqslant \frac{1 + 2^{\alpha+1}\alpha\varepsilon f_\infty}{1 - 2^{\alpha+1}\alpha\varepsilon f_\infty} \|c_0^N\|_\alpha^2 + \frac{K_\alpha}{1 - 2^{\alpha+1}\alpha\varepsilon f_\infty} \\
\times \int_0^t \mathbb{E}[\|c^{\varepsilon,N}\|_\alpha^2(t' \wedge T^\varepsilon)]\, dt' \qquad (5.24)
\end{aligned}$$

so that we have for any $\varepsilon < (2^{\alpha+2}\alpha f_\infty)^{-1}$:

$$\mathbb{E}[\|c^{\varepsilon,N}(t \wedge T^\varepsilon)\|_\alpha^2] \leqslant 3\, \|c_0^N\|_\alpha^2 + 2K_\alpha \int_0^t \mathbb{E}[\|c^{\varepsilon,N}\|_\alpha^2(t' \wedge T^\varepsilon)]\, dt' \qquad (5.25)$$

Applying Gronwall's lemma completes the proof of the first step of the lemma, since $\|c_0^N\|_\alpha$ is uniformly bounded by $\|c_0\|_\alpha$.

Step 2: For any $t > 0$, there exists a function $K_{\alpha,t}(k)$ which goes to 0 as $k \to \infty$ such that, for any family of $\mathscr{F}^\varepsilon$-adapted stopping times T^ε:

$$\sup_{N \in \mathbb{N}} \sup_{\varepsilon \in (0,1)} \mathbb{E}\left[\sum_{p=k}^{\infty} (1+p)^\alpha |c_p^{\varepsilon,N}|^2(t \wedge T^\varepsilon)\right] \leqslant K_{\alpha,t}(k) \qquad (5.26)$$

For any $c \in l_\alpha^2$ we denote $\|c\|_{\alpha,k}^2 := \sum_{p=k}^\infty (1+p)^\alpha |c_p|^2$. Note that $\|c\|_{\alpha,k} \leqslant \|c\|_{\alpha,k-1} \leqslant \cdots \leqslant \|c\|_{\alpha,0} := \|c\|_\alpha$. From the martingale property of $M_p^{\varepsilon,N}$ and the estimates (5.10), (5.11), (5.12) and (5.13), we have for any $\varepsilon < (2^{\alpha+2}\alpha f_\infty)^{-1}$:

$$\mathbb{E}[\|c^{\varepsilon,N}\|_{\alpha,k}^2(t \wedge T^\varepsilon)] \leqslant 3\, \|c_0\|_{\alpha,k-1}^2 + 2K_\alpha \int_0^t \mathbb{E}[\|c^{\varepsilon,N}\|_{\alpha,k-2}^2(t' \wedge T^\varepsilon)]\, dt' \qquad (5.27)$$

Iterating this inequality we get that, for any $k \geqslant 2k_0 + 1$:

$$\begin{aligned}
&\mathbb{E}[\, \|c^{\varepsilon, N}\|^2_{\alpha, k}\, (t \wedge T^\varepsilon)] \\
&\quad \leqslant 3\, \|c_0\|^2_{\alpha, k-1} + 6K_\alpha t\, \|c_0\|^2_{\alpha, k-3} + \cdots + 3\, \frac{(2K_\alpha t)^{k_0}}{k_0!}\, \|c_0\|^2_{\alpha, k-2k_0-1} \\
&\quad + \frac{(2K_\alpha t)^{k_0}}{k_0!} \sup_{s \in [0, t]} \mathbb{E}[\, \|c^{\varepsilon, N}(s \wedge T^\varepsilon)\|^2_\alpha] \qquad (5.28)
\end{aligned}$$

Applying (5.19) then establishes that, for any $\varepsilon < (2^{\alpha+2}\alpha f_\infty)^{-1}$ and for any N:

$$\mathbb{E}[\, \|c^{\varepsilon, N}\|^2_{\alpha, k}\, (t \wedge T^\varepsilon)] \leqslant 3e^{2K_\alpha t}\, \|c_0\|^2_{\alpha, k-2k_0-1} + \frac{(2K_\alpha t)^{k_0} K_{\alpha, t}}{k_0!} \qquad (5.29)$$

Let us fix $t \geqslant 0$. Let $\eta > 0$. On the one hand there exists some k_0 such that:

$$\frac{(2K_\alpha t)^{k_0} K_{\alpha, t}}{k_0!} \leqslant \frac{\eta}{2} \qquad (5.30)$$

On the other hand there exists some k_1 such that, for any $k \geqslant k_1$:

$$\|c_0\|^2_{\alpha, k} \leqslant \frac{\eta}{6}\, e^{-2K_\alpha t} \qquad (5.31)$$

Consequently, for any $k \geqslant k_1 + 2k_0 + 1$, we have $\mathbb{E}[\, \|c^{\varepsilon, N}\|^2_{\alpha, k}\, (t \wedge T^\varepsilon)] \leqslant \eta$, which yields (5.26).

Step 3: For any $\delta > 0$, $t > 0$, there exists a function $K_{\delta, \alpha, t}(N)$ which goes to 0 as $N \to \infty$ such that:

$$\sup_{N' \in \mathbb{N}} \sup_{\varepsilon \in (0, 1)} \mathbb{P}\, (\sup_{s \in [0, t]} \|c^{\varepsilon, N+N'}(s) - c^{\varepsilon, N}(s)\|_\alpha \geqslant \delta) \leqslant K_{\delta, \alpha, t}(N) \qquad (5.32)$$

Let us fix some $T_0 > 0$ and $N' \geqslant 3$. Using the perturbed test function method as in Step 1, we show that, for any $p = 0, \ldots, N-1, N+2, \ldots, N+N'-1$:

$$\begin{aligned}
&M_p^{\varepsilon, N, N'}(t) \\
&\quad := |c_p^{\varepsilon, N+N'} - c_p^{\varepsilon, N}|^2\, (t) - |c_0^{N+N'}{}_p - c_0^N{}_p|^2 \\
&\quad + \varepsilon \sum_{j=1}^{2} ((J_j(c^{\varepsilon, N+N'} - c^{\varepsilon, N}))_p\, (t)\, f_j^\varepsilon(t) - J_j(c_0^{N+N'} - c_0^N))_p\, f_j^\varepsilon(0)) \\
&\quad - \sum_{j=1}^{2} \int_0^t (H_j^\varepsilon(c^{\varepsilon, N+N'} - c^{\varepsilon, N}))_p\, (t')\, m^\varepsilon(t')\, f_j^\varepsilon(t')\, dt' \qquad (5.33)
\end{aligned}$$

is a $\mathscr{F}^\varepsilon$-martingale. For $p=N$:

$$M_N^{\varepsilon, N, N'}(t) + (N+1) \sum_{j=1}^{2} \int_0^t \operatorname{Re}(c_N^{\varepsilon, N*}(c_N^{\varepsilon, N+N'} - c_N^{\varepsilon, N}))(t') \times m^\varepsilon(t')\, \xi_j^\varepsilon(t')\, f_j^\varepsilon(t')\, dt' \tag{5.34}$$

is a $\mathscr{F}^\varepsilon$-martingale. For $p=N+1$:

$$M_{N+1}^{\varepsilon, N, N'}(t) + (N+1) \sum_{j=1}^{2} \int_0^t \operatorname{Re}(c_N^{\varepsilon, N*}(c_N^{\varepsilon, N+N'} - c_N^{\varepsilon, N}))(t') \times m^\varepsilon(t')\, \xi_j^\varepsilon(t')\, f_j^\varepsilon(t')\, dt' \tag{5.35}$$

is a $\mathscr{F}^\varepsilon$-martingale. For $p=N+N'$:

$$M_{N+N'}^{\varepsilon, N, N'}(t) - (N+N'+1) \sum_{j=1}^{2} \int_0^t |c_{N+N'}^{\varepsilon, N+N'}|^2\,(t')\, m^\varepsilon(t')\, \xi_j^\varepsilon(t')\, f_j^\varepsilon(t')\, dt' \tag{5.36}$$

is a $\mathscr{F}^\varepsilon$-martingale. Let T^ε be a $\mathscr{F}^\varepsilon$-adapted stopping time such that $T^\varepsilon \leqslant T_0$ almost surely. We then get from (5.10), (5.11), (5.12) and (5.13) that, for any $\varepsilon < (2^{\alpha+2}\alpha f_\infty)^{-1}$:

$$\begin{aligned} &\mathbb{E}[\,\|c^{\varepsilon, N+N'} - c^{\varepsilon, N}\|_\alpha^2\,(t \wedge T^\varepsilon)] \\ &\quad\leqslant 3\,\|c_0^{N+N'} - c_0^N\|_\alpha^2 + 2K_\alpha \int_0^t \mathbb{E}[\,\|c^{\varepsilon, N+N'} - c^{\varepsilon, N}\|_\alpha^2\,(t' \wedge T^\varepsilon)]\, dt' \\ &\qquad + 2\alpha C_0(2+N)^\alpha \int_0^t \mathbb{E}[\,|c_N^{\varepsilon, N}|\,|c_N^{\varepsilon, N+N'} - c_N^{\varepsilon, N}|\,(t' \wedge T^\varepsilon)]\, dt' \end{aligned} \tag{5.37}$$

Applying the inequality $ab \leqslant \frac{1}{2}a^2 + \frac{1}{2}b^2$ and (5.26) in the case $k=N$ to the last term of the right-hand side of this inequality yields:

$$\begin{aligned} &\mathbb{E}[\,\|c^{\varepsilon, N+N'} - c^{\varepsilon, N}\|_\alpha^2\,(t \wedge T^\varepsilon)] \\ &\quad\leqslant 3\,\|c_0^{N+N'} - c_0^N\|_\alpha^2 + 2K'_\alpha \int_0^t \mathbb{E}[\,\|c^{\varepsilon, N+N'} - c^{\varepsilon, N}\|_\alpha^2\,(t' \wedge T^\varepsilon)]\, dt' \\ &\qquad + K_{\alpha, t}(N) \times \alpha 2^\alpha C_0 \end{aligned} \tag{5.38}$$

where $K'_\alpha = K_\alpha + \alpha 2^\alpha C_0$. On the one hand, $K_{\alpha, t}(N)$ goes to zero as $N \to \infty$. On the other hand, by definition of c_0^N:

$$\|c_0^{N+N'} - c_0^N\|_\alpha^2 = \sum_{p=N+1}^{N+N'} (1+p)^\alpha |c_{0_p}|^2 \leqslant \sum_{p=N+1}^{\infty} (1+p)^\alpha |c_{0_p}|^2 = \|c_0 - c_0^N\|_\alpha^2 \tag{5.39}$$

goes to zero as $N \to \infty$ uniformly with respect to N'. Applying Gronwall's lemma consequently establishes:

$$\limsup_{N \to \infty} \sup_{\varepsilon \in (0,1)} \sup_{N' \in \mathbb{N}} \mathbb{E}[\|c^{\varepsilon, N+N'}(T^\varepsilon) - c^{\varepsilon, N}(T^\varepsilon)\|_\alpha^2] = 0 \tag{5.40}$$

If we take

$$T^\varepsilon = \inf\{s \leqslant T_0, \|c^{\varepsilon, N+N'}(s) - c^{\varepsilon, N}(s)\|_\alpha^2 \geqslant \delta\} \tag{5.41}$$

then we establish the statement of the third step.

Step 4: There exists a unique solution c^ε in $\mathbf{C}([0, \infty), l_\alpha^2)$ of (3.4), which also satisfies (5.4).

We can extract a subsequence $\phi(N)$ such that:

$$\mathbb{P}(\sup_{s \in [0,t]} \|c^{\varepsilon, \phi(N)}(s) - c^{\varepsilon, \phi(N+1)}(s)\|_\alpha \geqslant 2^{-N}) \leqslant 2^{-N} \tag{5.42}$$

By Borel–Cantelli's lemma, for almost every ω, there exists $N(\omega)$ such that $N \geqslant N(\omega)$ implies that:

$$\sup_{s \in [0,t]} \|c^{\varepsilon, \phi(N)}(s) - c^{\varepsilon, \phi(N+1)}(s)\|_\alpha \leqslant 2^{-N} \tag{5.43}$$

Therefore, as $N \to \infty$, $c^{\varepsilon, \phi(0)} - c^{\varepsilon, \phi(N)} = \sum_{j=0}^{N-1} c^{\varepsilon, \phi(j)} - c^{\varepsilon, \phi(j+1)}$ converge in $\mathbf{C}([0, t], l_\alpha^2)$ to some element of this space that we express in the form $c^{\varepsilon, \phi(0)} - c^\varepsilon$. Since A and B_j^ε are continuous from l_α^2 into $l_{\alpha-1}^2$, c^ε is found to be a strong solution of (3.4) in l^2. Letting $N \to \infty$ in (5.19), we get that c^ε belongs to l_α^2 and satisfies (5.4). Besides, if c_1^ε and c_2^ε are two solutions in l_α^2, then Step 1 establishes that $\|c_1^\varepsilon - c_2^\varepsilon\|_\alpha (t) = 0$ for every t, which yields uniqueness. The second point of the lemma is proved by letting $N' \to \infty$ in (5.32). The third and last point is then straightforward since $\mathrm{Re}\langle B^\varepsilon c, c\rangle = 0$ for any $c \in l_1^2$. ▮

5.2. The Limit Process

We consider the system of linear differential equations (3.7) starting from $c(0)=c_0$:

$$dc=\sqrt{\gamma}\,B_1c\,dW_{1t}+\sqrt{\gamma}\,B_2c\,dW_{2t}+\gamma Ac\,dt \tag{5.44}$$

We also denote by $\tilde{c}^N\in\mathbf{C}([0,\infty),\mathbb{C}^{N+1})$ the processes which are defined as the solutions of the finite-dimensional linear systems:

$$\begin{aligned}d\tilde{c}^N&=\sqrt{\gamma}\,\mathbf{F}_1^N\tilde{c}^N\,dW_{1t}+\sqrt{\gamma}\,\mathbf{F}_2^N\tilde{c}^N\,dW_{2t}+\gamma\mathbf{G}^N\tilde{c}^N\,dt\\ \tilde{c}_p^N(0)&=c_{0_p},\qquad \text{for}\quad 0\leqslant p\leqslant N\end{aligned} \tag{5.45}$$

where the linear mappings $\mathbf{F}_j^N$ and $\mathbf{G}^N$ are defined by $\mathbf{F}_j^N=\Pi_N\circ B_j\circ\Pi_N^{-1}$ and $\mathbf{G}_j^N=\Pi_N\circ A\circ\Pi_N^{-1}$ respectively. We finally introduce the auxiliary processes $c^N\in\mathbf{C}([0,\infty),l_\alpha^2)$ given by $c_p^N=\tilde{c}_p^N$ for $p\leqslant N$ and 0 otherwise.

Lemma 5.4. 1. If $c_0\in l_\alpha^2$, $\alpha\geqslant 2$, then there exists a unique solution c in $\mathbf{C}([0,\infty),l_\alpha^2)$ of the system (5.44). Moreover, for any $t\geqslant 0$, there exists a constant $K_{\alpha,t}$ such that:

$$\mathbb{E}[\|c(t)\|_\alpha^2]\leqslant K_{\alpha,t} \tag{5.46}$$

2. For any $\delta>0$, $t>0$, there exists a function $K_{\delta,\alpha,t}(N)$ which goes to 0 as $N\to\infty$ such that:

$$\mathbb{P}(\sup_{s\in[0,t]}\|c^N(s)-c(s)\|_\alpha\geqslant\delta)\leqslant K_{\delta,\alpha,t}(N) \tag{5.47}$$

Proof. In the following we work with the natural filtration generated by the σ-algebra $\mathscr{F}_s^t=\sigma(W_u^1,W_u^2,s\leqslant u\leqslant t)$. The proof is similar to that of Lemma 5.2.

Step 1: There exists a constant k_α such that, for any stopping time T:

$$\sup_{N\in\mathbb{N}}\mathbb{E}[\|c^N\|_\alpha^2(t\wedge T)]\leqslant\|c_0\|_\alpha^2\,e^{k_\alpha\gamma t} \tag{5.48}$$

For every $p\leqslant N-1$

$$M_p^N(t):=|c_p^N|^2(t)-|c_{0_p}|^2-2\gamma\int_0^t(h_1c^N)_p(t')\,dt' \tag{5.49}$$

is a $\mathscr{F}$-martingale, where h_1 has been defined in (5.6). For $p = N$, $M_N^N(t) - 2\gamma(N+1)\int_0^t |c_N^N|^2(t')\,dt$ is a $\mathscr{F}$-martingale. Let T be a stopping time. We then get from (5.11) and (5.13) that:

$$\mathbb{E}[\|c^N\|_\alpha^2(t \wedge T)] \leqslant \|c_0^N\|_\alpha^2 + (3\alpha^2 2^\alpha + 2^{\alpha+1}\alpha)\,\gamma \int_0^t \mathbb{E}[\|c^N\|_\alpha^2(t' \wedge T)]\,dt' \tag{5.50}$$

which yields the desired result since $\|c_0^N\|_\alpha \leqslant \|c_0\|_\alpha$.

Step 2: For any $\delta > 0$, $t > 0$ there exists a function $K_{\delta,\alpha,t}(N)$ which goes to 0 as $N \to \infty$ such that:

$$\sup_{N' \in \mathbb{N}} \mathbb{P}\big(\sup_{s \in [0,t]} \|c^N(s) - c^{N+N'}(s)\|_\alpha \geqslant \delta\big) \leqslant K_{\delta,\alpha,t}(N) \tag{5.51}$$

For any $p = 0,\ldots, N-1, N+2,\ldots, N+N'-1$:

$$\begin{aligned} M_p^{N,N'}(t) := {} & |c_p^{N+N'} - c_p^N|^2(t) - |c_{0_p}^{N+N'} - c_{0_p}^N|^2 \\ & - 2\gamma \int_0^t (h_1(c^{N+N'} - c^N))_p(t')\,dt' \end{aligned} \tag{5.52}$$

is a $\mathscr{F}$-martingale. For $p = N, N+1, N+N'-1$ we must add some corrections. The proof is then very similar to that of Lemma 5.2.

Step 3: There exists a unique solution c in $\mathbf{C}([0,\infty), l_\alpha^2)$ of the system (5.44), which also satisfies (5.46).

The proof is very similar as in the Step 3 of Lemma 5.2. We can extract a subsequence such that $c^{\phi(0)} - c^{\phi(N)} = \sum_{j=0}^{N-1} c^{\phi(j)} - c^{\phi(j+1)}$ converge in $\mathbf{C}([0,t], l_\alpha^2)$ to some element of this space that we express in the form $c^{\phi(0)} - c$. Since A and B_j are continuous from l_α^2 into $l_{\alpha-2}^2$ (we have to assume that $\alpha \geqslant 2$) c is found to be a strong solution of (3.4) in l^2. The end of the proof is the same as the one of Lemma 5.2. ▮

Lemma 5.5. For any N, the processes $c^{\varepsilon,N} \in \mathbf{C}([0,\infty), l_\alpha^2)$ converge in distribution to the process $c^N \in \mathbf{C}([0,\infty), l_\alpha^2)$ as $\varepsilon \to 0$.

Proof. Applying Theorem IV-6-7 in ref. 18, the processes $\tilde{c}^{\varepsilon,N} \in \mathbf{C}([0,\infty), \mathbb{C}^{N+1})$ governed by the system (5.3) converge in distribution to the process $\tilde{c}^N \in \mathbf{C}([0,\infty), \mathbb{C}^{N+1})$ solution of the system (5.45). To be very rigorous, we need to separate the real and imaginary parts, so that we actually deal with processes in $\mathbf{C}([0,\infty), \mathbb{R}^{2(N+1)})$. The corresponding system then fulfils the hypotheses of Theorem IV-6-7 in ref. 18, because m is ϕ-mixing with $\phi \in L^{1/2}(\mathbb{R}_+)$. This yields the result. ▮

5.3. Tightness

We begin by stating some standard tightness criteria.[19]

Lemma 5.6. Let (E, d) be a Polish space, X^ε a process with paths in $\mathbf{D}([0, T_0], E)$. If for every t in a dense subset of $[0, T_0]$ the family $(X^\varepsilon(t))_{\varepsilon \in (0,1)}$ is tight in E and X^ε satisfies the Aldous property.

For any $\eta > 0$, $\lambda > 0$, there exists $\delta > 0$ such that

$$[\mathrm{A}] \quad \limsup_{\varepsilon \to 0} \sup_{T} \sup_{0<\theta<\delta} \mathbb{P}(d(X^\varepsilon(T+\theta), X^\varepsilon(T)) > \lambda) < \eta$$

where T is a stopping time and $\sup_T$ is the sup over all such $T \leqslant T_0 - \delta$, then the family $(X^\varepsilon)_{\varepsilon \in (0,1)}$ is tight in $\mathbf{D}([0, T_0]), E)$.

If the processes X^ε are continuous, then [A] is necessary for tightness. ∎

Lemma 5.7. Let H be a separable Hilbert space and H_N be an increasing sequence of finite-dimensional spaces in H such that, for any $h \in H$, $\lim_{N \to \infty} \pi_{H_N} h = h$. Let Y^ε be a H-valued process. Y^ε is tight if and only if for any $\eta > 0$ and $\lambda > 0$, there exists ρ_η and a subspace $H_{N(\eta, \lambda)}$ such that

$$\sup_{\varepsilon \in (0,1)} \mathbb{P}(\| Y^\varepsilon \| \geqslant \rho_\eta) \leqslant \eta \qquad \text{and} \qquad \sup_{\varepsilon \in (0,1)} \mathbb{P}(d(Y^\varepsilon, H_{N(\eta, \lambda)}) > \lambda) \leqslant \eta \tag{5.53}$$

Proposition 5.8. The process c^ε is tight in $\mathbf{D}([0, \infty), l^2_\alpha)$.

Proof. We shall prove on the one hand that $c^\varepsilon(t)$ is tight in l^2_α for any t and on the other hand that c^ε satisfies the Aldous property, which establishes the result by Lemma 5.6.

Step 1: $c^\varepsilon(t)$ is tight in l^2_α for any $t \geqslant 0$.

From Lemma 5.2, we have:

$$\sup_{\varepsilon \in (0,1)} \mathbb{P}(\|c^\varepsilon(t)\|_\alpha \geqslant M) \leqslant \frac{K_{\alpha, t}}{M^2} \tag{5.54}$$

Since $c_p^{\varepsilon, N} = 0$ for $p \geqslant N+1$, we have

$$d_\alpha(c^\varepsilon(t), H_N)^2 = \sum_{p=N+1}^{\infty} (1+p)^\alpha \, |c_p^\varepsilon(t)|^2 \leqslant \|c^\varepsilon(t) - c^{\varepsilon, N}(t)\|_\alpha^2 \tag{5.55}$$

which implies that $c^\varepsilon(t)$ is tight in l^2_α by Lemmas 5.2 and 5.7.

Step 2: c^ε satisfies the Aldous property.

Let $T_0 > 0$ and $\lambda > 0$. Applying Lemma 5.2, we have for any $\theta \leqslant 1$, for any stopping time $T \leqslant T_0 - \theta$ and for any $\varepsilon \in (0, 1)$:

$$\mathbb{P}(\|c^\varepsilon(T+\theta) - c^\varepsilon(T)\|_\alpha \geqslant \lambda) \leqslant \mathbb{P}(\|c^{\varepsilon, N}(T+\theta) - c^{\varepsilon, N}(T)\|_\alpha \geqslant \lambda/2) + 2K_{\lambda/4, \alpha, T_0}(N) \tag{5.56}$$

Let η be some positive number. We choose some N_0 such that $K_{\lambda/4, \alpha, T_0}(N_0) \leqslant \eta/4$. From Lemma 5.5, c^{ε, N_0} is tight in $\mathbf{C}([0, T_0], l^2_\alpha)$, so that there exists some $\delta > 0$ such that:

$$\limsup_{\varepsilon \to 0} \sup_T \sup_{0<\theta<\delta} \mathbb{P}(\|c^{\varepsilon, N_0}(T+\theta) - c^{\varepsilon, N_0}(T)\|_\alpha \geqslant \lambda/2) \leqslant \eta/2 \tag{5.57}$$

which finally establishes that

$$\limsup_{\varepsilon \to 0} \sup_T \sup_{0<\theta<\delta} \mathbb{P}(\|c^\varepsilon(T+\theta) - c^\varepsilon(T)\|_\alpha \geqslant \lambda) \leqslant \eta \tag{5.58}$$

5.4. Convergence of the Finite-Dimensional Distributions

Proposition 5.9. The finite-dimensional distributions of $(c^\varepsilon(t))_{t\geqslant 0}$ converge in distribution in l^2_α to the corresponding ones of $(c(t))_{t\geqslant 0}$.

Proof. Let us fix some $m \in \mathbb{N}$. Let F be a continuous and bounded function from $l^{2\otimes m}_\alpha$ into $\mathbb{R}$. Let $t_1 < \cdots < t_m$ be m nonnegative real numbers. We want to prove the weak convergence of $F(c^\varepsilon(t_1),..., c^\varepsilon(t_m))$ (that we denote by $F(c^\varepsilon)$) towards $F(c(t_1),..., c(t_m))$ (that we denote by $F(c)$). Expanding the difference $\Delta^\varepsilon := |\mathbb{E}[F(c^\varepsilon)] - \mathbb{E}[F(c)]|$ as follows:

$$\Delta^\varepsilon \leqslant |\mathbb{E}[F(c^\varepsilon)] - \mathbb{E}[F(c^{\varepsilon, N})]| + |\mathbb{E}[F(c^{\varepsilon, N})] - \mathbb{E}[F(c^N)]| + |\mathbb{E}[F(c^N)] - \mathbb{E}[F(c)]| \tag{5.59}$$

and estimating the first and third terms of the right-hand member, we obtain that, for any $M > 0$:

$$\begin{aligned}\Delta^\varepsilon \leqslant 2 \sup_{|c| \leqslant M,\, |c-c'| \leqslant \delta} &|F(c) - F(c')| + |\mathbb{E}[F(c^{\varepsilon, N})] - \mathbb{E}[F(c^N)]| \\ &+ 2\, \|F\|_\infty\, \mathbb{P}(\sup_{t<t_m} \|c^\varepsilon - c^{\varepsilon, N}\|_\alpha \geqslant \delta) + 2\, \|F\|_\infty\, \mathbb{P}(\sup_{t<t_m} \|c - c^N\|_\alpha \geqslant \delta) \\ &+ 2\, \|F\|_\infty \sum_{j=1}^m \mathbb{P}(\|c(t_j)\|_\alpha \geqslant M) + 2\, \|F\|_\infty \sum_{j=1}^m \mathbb{P}(\|c^\varepsilon(t_j)\|_\alpha \geqslant M)\end{aligned} \tag{5.60}$$

Taking the limsup as $\varepsilon \to 0$ and using Lemmas 5.2, 5.4 and 5.5,

$$\limsup_{\varepsilon \to 0} \Delta^{\varepsilon} \leqslant 2 \sup_{|c| \leqslant M,\, |c-c'| \leqslant \delta} |F(c) - F(c')| + 4 \|F\|_{\infty} K_{\delta, \alpha, t_m}(N) + 4m \|F\|_{\infty} \frac{K_{\alpha, t_m}}{M^2} \tag{5.61}$$

Taking first the limit $N \to \infty$ in the right-hand member, then the limit $\delta \to 0$, and finally $M \to \infty$, we establish the desired result. ∎

APPENDIX A: PROOF OF PROPOSITION 2.2

From the recursion relations amongst the Hermite polynomials,[20] we get that:

$$\begin{aligned}
\frac{\partial f_{p,q,r}}{\partial x} &= -\frac{\sqrt{p+1}}{\sqrt{2}} f_{p+1,q,r} + \frac{\sqrt{p}}{\sqrt{2}} f_{p-1,q,r} \\
x f_{p,q,r} &= \frac{\sqrt{p+1}}{\sqrt{2}} f_{p+1,q,r} + \frac{\sqrt{p}}{\sqrt{2}} f_{p-1,q,r} \\
\frac{\partial f_{p,q,r}}{\partial y} &= -\frac{\sqrt{q+1}}{\sqrt{2}} f_{p,q+1,r} + \frac{\sqrt{q}}{\sqrt{2}} f_{p,q-1,r} \\
y f_{p,q,r} &= \frac{\sqrt{q+1}}{\sqrt{2}} f_{p,q+1,r} + \frac{\sqrt{q}}{\sqrt{2}} f_{p,q-1,r} \\
\frac{\partial f_{p,q,r}}{\partial z} &= -\frac{\sqrt{r+1}}{\sqrt{2}} f_{p,q,r+1} + \frac{\sqrt{r}}{\sqrt{2}} f_{p,q,r-1} \\
z f_{p,q,r} &= \frac{\sqrt{r+1}}{\sqrt{2}} f_{p,q,r+1} + \frac{\sqrt{r}}{\sqrt{2}} f_{p,q,r-1}
\end{aligned} \tag{A.1}$$

If $\psi = \sum c_{p,q,r} f_{p,q,r}$, then we have:

$$\begin{aligned}
\|c\|_{\alpha}^2 &\leqslant \sum_{p,q,r=0}^{\infty} \left(\frac{(p+\alpha)!}{p!} + \frac{(q+\alpha)!}{q!} + \frac{(r+\alpha)!}{r!} \right) |c_{p,q,r}|^2 \\
&= \frac{1}{2^{\alpha}} \sum_{\zeta \in \{x,y,z\}} \left\| \left(\zeta - \frac{\partial}{\partial \zeta} \right)^{\alpha} \psi \right\|^2
\end{aligned} \tag{A.2}$$

Expanding $(\zeta - \partial/\partial\zeta)^\alpha$ and using Minkowski's inequality proves that: $\|c\|_\alpha^2 \leqslant \|\psi\|_\alpha^2$. On the other hand,

$$\|\psi\|_\alpha^2 = \frac{1}{2^\alpha} \sum_{\zeta \in \{x, y, z\}} \sum_{\delta_1, ..., \delta_\alpha \in \{0, 1\}} \|((-1)^{\delta_\alpha} \theta_{-1, \zeta} + \theta_{1, \zeta}) \circ \cdots \circ ((-1)^{\delta_1} \theta_{-1, \zeta} + \theta_{1, \zeta}) c\|^2 \tag{A.3}$$

where $(\theta_{1, x} c)_{p, q, r} := \sqrt{p+1}\, c_{p+1, q, r}$, and $(\theta_{-1, x} c)_{p, q, r} := \sqrt{p}\, c_{p-1, q, r}$ and $\theta_{j, y}$ (resp. $\theta_{j, z}$) acts on the q-index (resp. r-index). Straightforward estimates then show that:

$$\|\psi\|_\alpha^2 \leqslant 2^{2\alpha} \sum_{p, q, r=0}^{\infty} \left(\frac{(p+\alpha)!}{p!} + \frac{(q+\alpha)!}{q!} + \frac{(r+\alpha)!}{r!} \right) |c_{p, q, r}|^2 \leqslant 2^{2\alpha} \alpha!\, \|c\|_\alpha^2 \tag{A.4}$$

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